A THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER ON A PERMEABLE PLATE

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We examine the laminar boundary layer on a permeable half-plane for the case in which the streamline of the outside flow is a parabola, and in which the velocity of suction or expulsion is proportional to $(x + \text{const})^{1/2}$. We have found the velocity profiles in the boundary layer, as well as the frictional stress distribution on the half-plane.

The solution of the complete equations for a three-dimensional laminar boundary layer is a rather complex problem. We are therefore interested in examining the simplest cases, using these as an example by means of which to determine certain significant features of flow in a three-dimensional boundary layer. One such case, studied by Loos [1],* involves the streamlining of a fixed half-plane case, studied by a "parabolic" external flow (the streamlines are parabolas). The simplicity of the problem permits us simultaneously to examine the interesting problem (from the practical standpoint) of the influence exerted by the suction or expulsion on the characteristics of a three-dimensional boundary layer. The attempt to solve the problem of the boundary layer on a permeable half-plane streamlined by a parabolic flow was undertaken by Kozlov [2]. However, in [2] the equations determining the velocity profile in the boundary layer were incorrectly derived. Below we solve the problem considered by Kozlov. In addition to solutions of the form given in [1] and [2], we have found yet another solution – a self-similar solution.

Let a semiinfinite flat plate (a half-plane) be streamlined by a flow of an incompressible viscous fluid. Let us examine the laminar boundary layer on the plate in the case in which the projections of the velocity of the external flow onto the axis of the coordinate system shown in Fig. 1 are determined by the formulas

$$U = \text{const}, \qquad W = a + bx, \tag{1}$$

where a and b are constants. The streamlines of this flow are parabolas:

$$2Uz = 2ax + bx^2 + \text{const}. \tag{2}$$

Parabolic flow is vortical, the vortex vector Ω is directed along the normal to the plate surface, so that $\Omega_y \doteq -b$, while the pressure varies only along the z-axis:

$$p = \text{const} - \rho bUz. \tag{3}$$

Because of the infinity of the plate in the z-direction and because the projections of the velocity of the external flow are independent of z, the velocity in the boundary layer, given appropriate conditions at the wall, are also independent of z. The boundary-layer equations for this case therefore assume the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{4}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = Ub + \nu\frac{\partial^2 w}{\partial y^2}, \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{6}$$

*W. Wuest [9] studied a special case of this problem.

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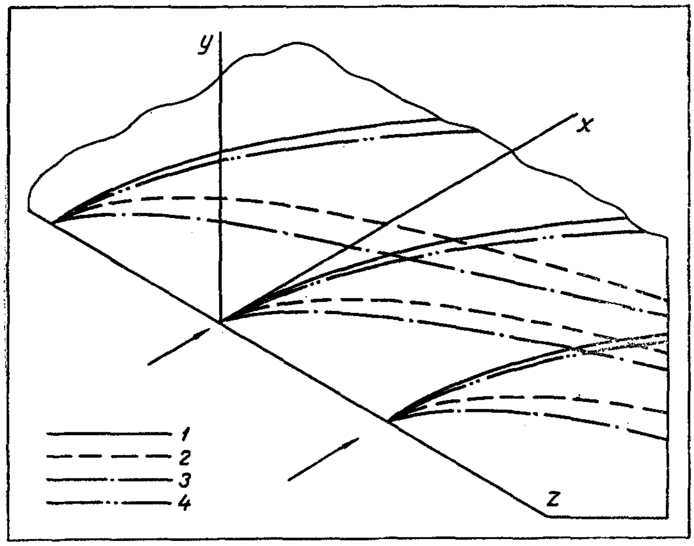

Fig. 1. Effect of suction or expulsion on the flow pattern in the boundary layer: 1) projections of the streamlines of the external flow onto the plane of the plate; 2) extreme streamlines for s = 0; 3) the same, for s = −0.5; 4) the same for s = 5.

It is assumed that the suction or expulsion of the liquid is accomplished through the plate surface along the normal to that surface at a velocity $v_0 = v_0(x)$. Then

$$u = 0,\ v = -v_0(x),\ w = 0 \text{ when } y = 0;$$
$$u = U,\ w = a + bx \quad \text{when } y = \infty, \tag{7}$$

with v_0 positive in the case of suction, and negative in the case of expulsion.

System (4)-(6) can be reduced to a system of ordinary differential equations, one of which is the Blasius equation. We can do this in the following two ways.

1. Let us seek the projections of the velocity in the boundary layer in the form

$$u = \frac{1}{2} UF'(\eta_1), \quad v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(F\eta_1 - F), \quad w = aw_0(\eta_1) + bxw_1(\eta_1),$$

where

$$\eta_1 = \frac{1}{2} y \sqrt{\frac{U}{\nu x}}, \quad v_0 = \frac{s}{2}\sqrt{\frac{\nu U}{x}}. \tag{8}$$

Then Eqs. (4)-(6) and conditions (7) will be satisfied for any values of the constants U, a, and b, if the functions F, w_0, and w_1 are solutions of the boundary-value problems:

$$F''' + FF'' = 0,\ F(0) = s,\ F'(0) = 0,\ F'(\infty) = 2, \tag{9}$$

$$w_0'' + Fw_0' = 0, \quad w_0(0) = 0, \quad w_0(\infty) = 1, \tag{10}$$

$$w_1'' + Fw_1' - 2F'w_1 = -4, \quad w_1(0) = 0, \quad w_1(\infty) = 1. \tag{11}$$

The constant s, determining the intensity of expulsion or suction, must be specified. Assuming in (8)-(11) that s = 0, we come to the problem solved by Loos [1], and if we assume that a = b = 0 we come to the two-dimensional problem solved by Emmons and Leigh [3].

2. If b ≠ 0 and the signs of the constants a and b coincide, system (4)-(6) has a self-similar solution (see the table of boundary conditions in [4]):

$$u = \frac{1}{2} UF'(\eta_2), \quad v = \frac{1}{2}\sqrt{\frac{\nu U}{x + x_0}}(F'\eta_2 - F), \quad w = b(x + x_0)\Phi(\eta_2), \tag{12}$$

TABLE 1. The Function $\Phi(\eta)$ for Various Values of s

η \ s	−1,20	−1,15	−1,10	−1,05	−1,00	−0,75	−0,5	−0,25
0	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,2	0,6003	0,5732	0,5553	0,5422	0,5319	0,5025	0,4905	0,4867
0,4	1,183	1,116	1,069	1,034	1,006	0,9168	0,8693	0,8415
0,6	1,743	1,619	1,533	1,467	1,413	1,237	1,138	1,074
0,8	2,276	2,077	1,939	1,832	1,744	1,462	1,304	1,204
1,0	2,775	2,481	2,278	2,122	1,996	1,597	1,383	1,254
1,2	3,535	2,824	2,644	2,332	2,163	1,651	1,395	1,249
1,4	3,647	3,097	2,731	2,460	2,248	1,639	1,361	1,213
1,6	4,004	3,295	2,830	2,508	2,257	1,679	1,300	1,165
1,8	4,298	3,413	2,862	2,480	2,199	1,490	1,231	1,117
2,0	4,522	3,440	2,812	2,387	2,087	1,389	1,165	1,077
2,2	4,665	3,401	2,694	2,244	1,940	1,291	1,110	1,047
2,4	4,726	3,280	2,523	2,068	1,774	1,204	1,069	1,026
2,6	4,700	3,094	2,316	1,876	1,607	1,135	1,040	1,014
2,8	4,583	2,858	2,091	1,687	1,453	1,084	1,022	1,007
3,0	4,392	2,590	1,866	1,513	1,321	1,049	1,011	1,003
3,2	4,124	2,309	1,657	1,364	1,216	1,027	1,005	1,001
3,4	3,795	2,035	1,476	1,246	1,137	1,014	1,002	1,000
3,6	3,426	1,984	1,320	1,157	1,082	1,007	1,001	
3,8	3,036	1,568	1,214	1,095	1,046	1,003	1,000	
4,0	2,649	1,392	1,133	1,054	1,025	1,001		
4,2	2,285	1,257	1,078	1,029	1,012	1,000		
4,4	1,961	1,160	1,043	1,014	1,006			
4,6	1,588	1,094	1,022	1,007	1,002			
4,8	1,448	1,052	1,011	1,003	1,001			
5,0	1,306	1,024	1,005	1,001	1,000			
5,2	1,180	1,017	1,002	1,000				
5,4	1,111	1,008	1,001					
5,6	1,081	1,003	1,000					
5,8	1,032	1,001						
6,0	1,016	1,000						
6,2	1,007							
6,4	1,003							
6,6	1,001							
6,8	1,000							

η \ s	0	1	2	3	4	5
0	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,1	0,2637	0,2916	0,3309	0,3753	0,4213	0,4668
0,2	0,4879	0,5187	0,5678	0,6219	0,6749	0,7237
0,3	0,6743	0,6904	0,7322	0,7790	0,8228	0,8608
0,4	0,8251	0,8161	0,8423	0,8757	0,9066	0,9318
0,5	0,9433	0,9045	0,9135	0,9334	0,9525	0,9676
0,6	1,032	0,9639	0,9575	0,9664	0,9769	0,9852
0,7	1,096	1,001	0,9832	0,9845	0,9893	0,9935
0,8	1,137	1,023	0,9971	0,9939	0,9954	0,9972
0,9	1,161	1,033	1,004	0,9983	0,9982	0,9989
1,0	1,171	1,036	1,006	1,0000	0,9994	0,9996
1,1	1,170	1,034	1,006		0,9999	0,9999
1,2	1,161	1,030	1,006		1,0000	1,0000
1,3	1,147	1,025	1,005			
1,4	1,130	1,020	1,003			
1,5	1,112	1,016	1,002			
1,6	1,094	1,012	1,002			
1,7	1,077	1,009	1,001			
1,8	1,062	1,006	1,001			
1,9	1,049	1,004	1,000			
2,0	1,038	1,003				
2,1	1,028	1,002				
2,2	1,021	1,001				
2,3	1,015	1,001				
2,4	1,011	1,000				
2,5	1,008					
2,6	1,005					
2,7	1,004					
2,8	1,002					
2,9	1,002					
3,0	1,001					
3,1	1,001					
3,2	1,000					

TABLE 1 (Continued)

η \ s	6	7	8	9	10	11	12
0	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,05	0,2979	0,3284	0,3582	0,3873	0,4154	0,4425	0,4685
0,10	0,5105	0,5520	0,5909	0,6271	0,6605	0,6912	0,7194
0,15	0,6610	0,7031	0,7409	0,7745	0,8040	0,8300	0,8527
0,20	0,7668	0,8045	0,8369	0,8644	0,8875	0,9069	0,9231
0,25	0,8406	0,8721	0,8980	0,9189	0,9358	0,9493	0,9601
0,30	0,8918	0,9168	0,9365	0,9518	0,9636	0,9726	0,9794
0,35	0,9270	0,9462	0,9608	0,9715	0,9795	0,9852	0,9894
0,40	0,9511	0,9655	0,9759	0,9833	0,9885	0,9921	0,9946
0,45	0,9675	0,9780	0,9853	0,9902	0,9936	0,9958	0,9973
0,50	0,9785	0,9860	0,9911	0,9943	0,9964	0,9978	0,9986
0,55	0,9859	0,9912	0,9946	0,9967	0,9980	0,9988	0,9993
0,60	0,9908	0,9945	0,9967	0,9981	0,9989	0,9994	0,9997
0,65	0,9941	0,9966	0,9981	0,9989	0,9994	0,9997	0,9998
0,70	0,9962	0,9979	0,9989	0,9994	0,9997	0,9998	0,9999
0,75	0,9976	0,9987	0,9993	0,9997	0,9998	0,9999	1,0000
0,80	0,9985	0,9992	0,9996	0,9998	0,9999	1,0000	
0,85	0,9991	0,9995	0,9998	0,9999	1,0000		
0,90	0,9994	0,9997	0,9999	0,9999			
0,95	0,9996	0,9998	0,9999	1,0000			
1,00	0,9998	0,9999	1,0000				
1,05	0,9999	0,9999					
1,10	0,9999	1,0000					
1,15	1,0000						

in which case

$$\eta_2 = \frac{1}{2} y \sqrt{\frac{U}{\nu(x+x_0)}}, \quad v_0 = \frac{s}{2}\sqrt{\frac{\nu U}{x+x_0}}, \quad x_0 = \frac{a}{b},$$

with the function $F(\eta_2)$ satisfying conditions (9), while the function $\Phi(\eta_2)$ is a solution for the boundary-value problem

$$\Phi'' + F\Phi' - 2F'\Phi = -4, \qquad \Phi(0) = 0, \qquad \Phi(\infty) = 1. \tag{13}$$

Comparison of (9), (10), and (11), (13) shows that

$$w_0 = \frac{1}{2} F'(\eta_1), \qquad w_1 = \Phi(\eta_1). \tag{14}$$

Consequently, solutions (8) and (12) are found simultaneously as a result of the solution of the boundary-value problems (9) and (13). The fact that the boundary value problem (4)-(7) has two solutions is a result not only of the difference between the velocities at the wall, but also a consequence of the difference "initial" conditions: solution (8) corresponds to a uniform velocity distribution for $x = 0$, while solution (12) corresponds to a nonuniform distribution. When $a = 0$, the two solutions coincide.

Despite the differences in the boundary conditions and in the method of finding the solutions, (8) and (12) are closely related to each other. Let us assume in formulas (8) that $a = 0$ and let us transfer the coordinate origin to the point $(x_1, 0, 0)$, $x_1 > 0$. In this new system of coordinates formulas (8) assume the same form as formulas (12), but in the place of x_0 we will have x_1. The quantity x_1 can be assumed to be equal to x_0 in solution (12). The flow which is then described by (12), beginning from the leading edge, coincides with the flow described by (8) for $a = 0$ and $x \geq x_0$. We will therefore not consider solution (12) in the following.

The boundary-value problem (9) has been solved in [3], where we find tables of the function F and its derivatives for various values of s. These tables can also be used for the calculation of the function Φ. The boundary-value problem (13) was solved by replacing the differential equation by a difference equation and subsequently applying a pivot method. The results of the calculation are shown in Table 1. It should be noted that there exists a limit value $s = s_\infty$ such that $F''(0) \to 0$ as $s \to s_\infty$. This limit value is not achieved, since the boundary-value problem (9) has no solution satisfying the condition $F''(0) = 0$. (Indeed, given the initial conditions $F(0) = s$, $F'(0) = F''(0) = 0$, the solution of the Blasius equation is $F = s = \text{const.}$) In other words, when $s \leq s_\infty$, there is no Prandtl boundary layer. According to the Emmons and Leigh calculations [3] we have $s_\infty \approx -1.23849$.

TABLE 2. Functions of the Parameter s, Determining the Frictional Stress at the Wall

s	α	β	k	s	α	β	k
—1,20	0,01343178	3,040	452,7	2	4,6776781	3,856	1,648
—1,15	0,03859712	2,933	152,0	3	6,5289066	4,549	1,393
—1,10	0,06911819	2,868	82,99	4	8,4295589	5,329	1,261
—1,05	0,10383601	2,821	54,36	5	10,359623	6,144	1,185
—1,00	0,14207803	2,789	39,26	6	12,308238	7,000	1,137
—0,75	0,37445574	2,718	14,52	7	14,269139	7,888	1,105
—0,50	0,65796369	2,723	8,278	8	16,238520	8,798	1,083
—0,25	0,97872869	2,767	5,655	9	18,213964	9,725	1,067
0	1,3282293	2,836	4,270	10	20,193871	10,66	1,055
0,5	2,0912914	3,026	2,893	11	22,177150	11,51	1,046
1	2,9154668	3,266	2,240	12	24,163033	12,57	1,039

The components τ_x and τ_z of the frictional stress at the wall, referred to the dynamic head $(1/2)\rho U^2$, are expressed by the formulas

$$\bar{\tau}_x = \frac{2\mu}{\rho U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\alpha}{2}\sqrt{\frac{\nu}{Ux}},$$
$$\bar{\tau}_z = \frac{2\mu}{\rho U^2}\left(\frac{\partial w}{\partial y}\right)_{y=0} = \frac{\alpha}{2}(k\bar{x}+\bar{x}_0)\sqrt{\frac{\nu}{Ux}}, \tag{15}$$

where

$$\bar{x} = \frac{bx}{U}, \quad \bar{x}_0 = \frac{bx_0}{U} = \frac{a}{U}, \quad k = \frac{2\beta}{\alpha}, \quad \alpha = F''(0), \quad \beta = \Phi'(0).$$

The quantities α, β, and k as functions of s are shown in Table 2. We see from the table that $\infty > k > 1$ when $s_\infty < s < \infty$. For a fixed value of s the frictional stress $\tau = \sqrt{\tau_x^2 + \tau_z^2}$ at the wall – infinite at the leading edge, diminishing with increasing x when $b \neq 0$ – reaches its minimum at the straight line $x = 1/k(1 + \bar{x}_0^2)^{1/2}$ and then again increases. Substituting (15) into the equation for the family of the friction lines

$$\frac{dx}{\tau_x} = \frac{dz}{\tau_z}$$

and integrating this equation, we find for $t \neq 0$ that the frictional stress at each point of the plate is directed along the tangent to the parabola

$$\bar{z} = \frac{k}{2}\left(\bar{x} + \frac{\bar{x}_0}{k}\right)^2 + \text{const} \quad \left(\bar{z} = \frac{bz}{U}\right), \tag{16}$$

passing through this point. When b = 0 the friction lines degenerate into straight lines $Uz = ax + \text{const}$. From (15) we determine the resistance-force components $0 \leq x \leq l$, $z_1 \leq z \leq z_2$ of the rectangular segment of the plate and the coefficients of frictional resistance:

$$C_{fx} = \frac{\alpha}{\sqrt{\text{Re}}}, \quad C_{fz} = \frac{\alpha}{\sqrt{\text{Re}}}\left(\frac{k}{3}\bar{l} + \bar{x}_0\right),$$

where

$$\text{Re} = \frac{Ul}{\nu}, \quad \bar{l} = \frac{bl}{U}.$$

Let us examine the flow pattern in the boundary layer. In the simplest case of b = 0, the external flow is uniform and inclined toward the leading edge; from (8) and (14) we have $w/u = W/U$. Consequently, the inclination of the leading plate edge toward the line of the freestream velocity exerts no influence on the development of the boundary layer, i. e., the problem reduces to the familiar [3] two-dimensional self-similar problem of the streamlining of a permeable plate by a uniform flow at a velocity $U_1 = \sqrt{U^2 + a^2}$. This situation, established for an impermeable plate in a gradient-free flow by Struminskii [5] and Sears [6], remains valid even in the case of fluid suction or expulsion through the plate surface at a velocity $v_0 \sim (x + \text{const})^{-1/2}$.

If $b \neq 0$, but $a = 0$, solution (8) will also be self-similar and the velocity profiles in the boundary layer will be expressed by the functions F' and Φ. The streamline of (2) in this case will be parabolas of the form

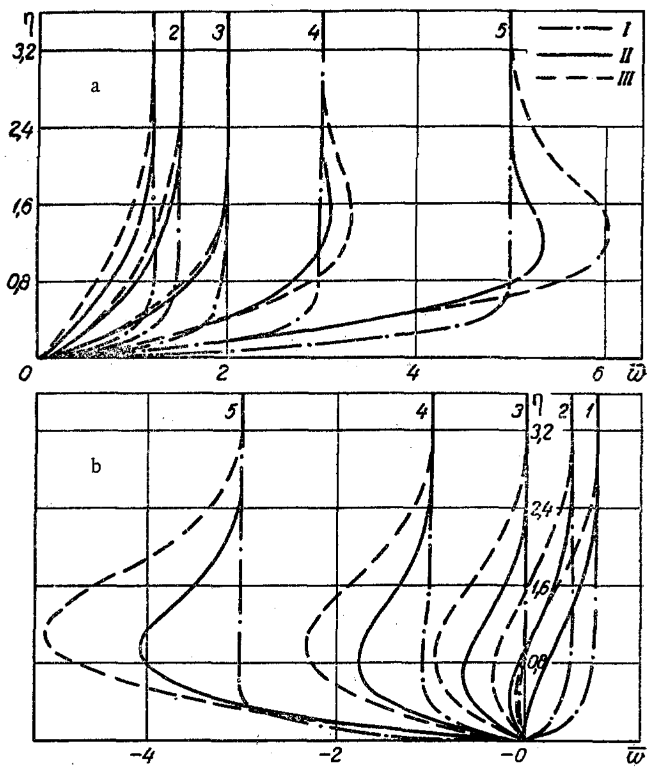

Fig. 2. Profiles of the velocity $\bar{w}$ for positive (a) and negative (b) values of $\bar{x}$ (I – s = 5; II – s = 0; III – s = –0.5; a: 1) x = 0.25; 2) 0.5; 3) 1; 4) 2; 5) 4; b: 1) x = –0.25; 2) (–0.5); 3) (–1); 4) (–2); 5) (–4).

$$\bar{z} = \frac{1}{2}\bar{x}^2 + \text{const.} \tag{17}$$

Near the plate surface the direction of the flow is determined by the extreme streamline which coincides with the friction lines, i.e., with the parabolas

$$\bar{z} = \frac{k}{2}\bar{x}^2 + \text{const.} \tag{18}$$

The coefficient $k > 1$ in Eq. (18) quantitatively characterizes the well-known fact that the streamlines in the boundary layer are bent more steeply than in the external layer. The more intense the suction ($1 < k < 4.27$), the closer the extreme streamline to the streamlines of the external flow. This is explained by the fact that those layers of the fluid which have been decelerated the most are removed from the flow, the velocity component that is parallel to the wall increases near the wall and, consequently, there is an increase in the centrifugal forces which act on the fluid particles moving along the bend trajectories. Since the pressure gradient does not change in this case (it is set by the external flow), the streamlines in the boundary layer are bent less. With expulsion ($4.27 < k < \infty$), conversely, the fluid enters the flow at a velocity which is equal to zero in the direction parallel to the wall. The fluid is seemingly decelerated and the slope of the streamlines increase (see Fig. 1). Because the slope of the streamline is greater in the boundary layer than in the external flow, the fluid particles pass through the point of the boundary layer that are situated along some vertical. These fluid particles move out of a region in which the pressure is greater than in the region from which the fluid particles come in the external flow, intersecting the same vertical [1]. This explains the "hump" in the profile of w for small s ($\Phi > 1$, beginning from some value of η) and it is further explained by the fact that the velocity of the external flow on that same vertical.

The effect of suction or expulsion on surface friction can be seen from (15) and Table 2. The component τ_x, as in the case of two-dimensional conditions, increases with suction and diminishes with expulsion.

The component τ_z also increases with suction (although more slowly), while expulsion results in virtually no change in the magnitude of this component. This is a consequence of the opposed effect of the two factors. On the one hand, expulsion reduces the inclination of the velocity profile at this point to the axis at the wall, i.e., it reduces friction. On the other hand, because of the increase in the slope of the streamline in the boundary layer, particles reach this point at a greater velocity and the velocity-profile inclination increases as a result, i.e., friction increases. When $s < -0.75$ the second factor becomes predominant.

In the general case $a \neq 0$, $b \neq 0$ the solution given in (8) will no longer be self-similar and the profiles for the projections of w at various distances from the leading edge will differ. Figure 2 shows the profiles of the dimensionless velocity $\bar{w} = w/a$ for several values of the dimensionless parameter $\bar{x} = bx/a$ and various s. The curves show that when $a/b > 0$ the velocity distribution in the boundary layer coincides qualitatively with the velocity distribution for $a = 0$. Again, with sufficient distance from the leading edge, we find a "hump" on the curves for $\bar{w}$, with the expulsion increasing this "hump," while suction diminishes it. When $a/b < 0$ the pattern is altered markedly. The component W of the velocity of the external flow changes direction, and near the leading edge this component is directed counter to the pressure gradient. Consequently, the profile for w, changing sign through the thickness of the layer, assumes the form which is found in two-dimensional flows beyond the separation point (see the profiles for $\bar{w}$ when $x = -0.5$ in Fig. 2b).

However, there is no separation of the boundary layer in the case of parabolic external flow. In the special cases considered above, as well as in the general case in which $a/b > 0$, there can be no separation, since the velocity vector in the boundary layer has no component at any point directed counter to the pressure gradient. If $a/b < 0$, such a component exists near the leading edge. However, from the location of the extreme streamlines, as shown in (16), we see that in this case there will be no separation of the boundary layer – the fluid will flow in the direction perpendicular to that of the pressure gradient. We can prove the absence of separation by employing the criteria for the separation of a three-dimensional laminar boundary layer (see [7] and [8]).

In conclusion, let us note that when $s \approx 1.033$ the "expulsive" effect of the boundary layer is offset by the suction and the component of velocity that is normal to the wall disappears at the outside edge of the boundary layer. This is the only value of s at which the boundary layer actually "comes into contact" with the external flow.

NOTATION

x, y, and z	are Cartesian coordinates (see Fig. 1);
u, v, and w	are the projections of the velocities in the boundary layer onto the x-, y-, and z-axes, respectively;
U and W	are the projections of the external-flow velocities onto the x- and z-axes, respectively;
v_0	is the suction for expulsion velocity;
p	is the pressure;
ρ	is the density;
μ	is the viscosity coefficient;
ν	is the kinetic coefficient of viscosity;
s	is the suction or expulsion parameter;
τ_x and τ_z	are the frictional-stress components at the wall;
C_{fx} and C_{fz}	are the coefficients of frictional resistance;
a and b	are constants in the expression for W;
$x_0 = a/b$;	
$\bar{x} = x/x_0$.	

LITERATURE CITED

1. H. C. Loos, JAS, 22, No. 1 (1955).
2. L. F. Kozlov, Prikl. Mekhn., AN USSR, Otd. Matematiki i Kibernetiki, II, No. 2 (1966).
3. H. W. Emmons and D. C. Leigh, ARC Current Papers, No. 157 (1954).
4. B. M. Berkovskii, Inzhern. Fiz. Zh., No. 1 (1964).

5. V. V. Struminskii, Collection of Theoretical Papers on Aerodynamics [in Russian], Oborongiz (1957).
6. W. R. Sears, JAS, 15, No. 1 (1948).
7. E. A. Eichelbrenner and A. Oudart, La Recherche Aeron., No. 40 (1954).
8. E. A. Eichelbrenner, ONERA Publ., No. 89 (1954).
9. W. Wuest, Bericht Max Planck Institut fur Stromungsforschung, No. 13 (1955).